Non-obtainable Continuous Functionals

by

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Abstract

For each $k \geq 3$ we construct a continuous functional Δ of type $k+1$ with a recursive associate such that Δ is not Kleene-computable in any continuous functional of type $\leq k$.

1. Introduction.

The countable or continuous functionals were first defined independently by Kleene [5] and Kreisel [6]. Kleene's countable functionals is a sub-class of the total functionals while Kreisel's continuous functionals are equivalence-classes of functions $f : \mathbb{N} \to \mathbb{N}$. In this paper we will regard the countable functionals as a type-structure $< Ct(k) >_{k\in\omega}$ where each $\psi \in Ct(k+1)$ is a total map $\psi : Ct(k) \to \omega$. This is equivalent to Kreisel's definition and it was also used in e.g. Bergstra [1] and Gandy - Hyland [3].

We will work with a fixed $k \geq 3$. We let n, m, k, i, j etc. denote natural numbers, $f, g, h, \alpha, \beta, \gamma$ will denote elements of $Ct(1)$, F will denote an element of $Ct(k-1)$, ϕ, ψ will denote elements of $Ct(k)$ and Δ will denote an element of $Ct(k+1)$.

We let $\sigma, \tau, \pi, \delta$ denote finite sequences which we without mentioning will identify with their sequence-numbers. $\sigma(n-1)$ will denote the n'th coordinate of σ when $0 < n \leq lh(\sigma)$. We use the standard notation $\bar{f}(n) = \langle f(0), \ldots ,f(n-1)\rangle$ and $\bar{\sigma}(n) = \langle\sigma(0), \ldots ,\sigma(n-1)\rangle$ whenever $n \leq lh(\sigma)$.

Kleene [5] showed that the class of countable functionals is closed under S1 - S9 (Kleene [4]), and he showed that all computable functionals are recursive, i.e. have recursive associates.

Later Tait showed that the converse is not true. The fan-functional Φ is recursive but not computable in any f . Φ is a functional working on two arguments $G \in Ct(2)$ and f. If

$$C_f = \{g \,;\, \forall n \; g(n) \leq f(n)\}$$

we let

$$\Phi(G,f) = \mu n \, \forall g_1, g_2 \in C_f \, (\bar{g}_1(n) = \bar{g}_2(n) \Rightarrow G(g_1) = G(g_2))$$

Tait never published his result, but sufficient arguments are given in e.g. Gandy - Hyland [3], Fenstad [2] and Normann [8] .

Later Gandy defined a new functional Γ in $Ct(3)$ as follows

$$\Gamma(G) = G_0(\lambda n \, \Gamma(G_{n+1}))$$

where

$$G_n(f) = G(n^* f) \qquad (^* \text{ denotes concatenation})$$

Gandy showed that Γ is recursive and Hyland showed that Γ is not computable in Φ and any f . The proof is based on some material in Bergstra [1] and can be found in Gandy - Hyland [3] and Normann [8] .

The following problem still remained open: "Are all continuous functionals computable in an element of $Ct(3)$?" In this paper we solve this problem by constructing a recursive $\Delta \in Ct(k+1)$ for all $k \geq 3$ such that Δ is not computable in any $\phi \in Ct(k)$.

2. Conventions and preliminaries.

From now on we will use the following notation and conventions:

Let B^n_σ denote the set of functionals in $Ct(n)$ with an associate extending σ. We will then have

$$B^1_\delta = \{f \,;\, \bar{f}(1h(\delta)) = \delta\}$$

When we use the letters σ and τ we will always assume that $B^{k-1}_\sigma \neq \emptyset$, $B^{k-1}_\tau \neq \emptyset$.

Lemma 1

a $B^{k-1}_\sigma \subseteq B^{k-1}_\tau \Leftrightarrow \forall\delta, s(\tau(\delta) = s+1 \Rightarrow \exists\pi\ B^{k-2}_\delta \subseteq B^{k-2}_\pi \wedge \sigma(\pi) = s+1\ )$

b If $B^{k-1}_\sigma \subseteq B^{k-1}_{\tau_1} \cup\ldots\cup B^{k-1}_{\tau_n}$ then $\exists i \leq n\ B^{k-1}_\sigma \subseteq B^{k-1}_{\tau_i}$

c If $B^{k-1}_\sigma \not\subseteq B^{k-1}_{\tau_1} \cup\ldots\cup B^{k-1}_{\tau_n}$ then there is an extension σ_1 of σ such that $B^{k-1}_{\sigma_1} \cap (B^{k-1}_{\tau_1} \cup\ldots\cup B^{k-1}_{\tau_n}) = \emptyset$.

Both this and the next lemma are elementary and we will not prove them here.

Lemma 2

a Let $I = \{\sigma;\ B^{k-1}_\sigma \neq \emptyset\}$. There is a primitive recursive family $\{F_\sigma\}_{\sigma\in I}$ in $Ct(k-1)$ such that

i $F_\sigma \in B^{k-1}_\sigma$

ii $F_\sigma = F_\tau \wedge \sigma \neq \tau \Rightarrow B^{k-1}_\sigma$ contains just F_σ which is constant.

iii If $\sigma<\tau$ and $B^{k-1}_\tau \not\subseteq B^{k-1}_\sigma$ then $F_\tau \notin B^{k-1}_\sigma$.

b There is a primitive recursive dense family $\{\xi_i\}_{i\in\mathbb{N}}$ in $Ct(k-2)$ such that the relation $\xi_i \in B^{k-2}_\delta$ is primitive recursive.

For each F we let $h_F(i) = F(\xi_i)$. The following result was essentially first proved in Normann [7] . Later S. Dvornickov simplified the proof. His proof is given in Normann [9] .

Lemma 3

a Let $H = \{h_F ;\ F \in Ct(k-1)\}$. Then $H \in \Pi^1_{k-2} \setminus \Sigma^1_{k-2}$.

b If A is Π^1_{k-1} then there is a primitive recursive R such that

$$\alpha \in A \Leftrightarrow \forall h \in H\ \exists n\ R(\bar{\alpha}(n), \bar{h}(n), n)$$

Definition

Let $G \in Ct(n)$, $n \geq 2$. We call α a semi-associate for G if $\forall m\ G \in B^n_{\bar{\alpha}(m)}$.

In proving the properties of Φ and Γ mentioned above we make use of the following observation:

If $G \in Ct(2)$ then a computation $\{e\}(G)$ depends only on G restricted to a countable set, namely $1\text{-sc}(G) = \{f ;\ f$ is computable in $G\}$.

So if α is a semi-associate for G securing all $f \in 1\text{-sc}(G)$ then there is an n such that $\{e\}(G)$ is uniquely determined by $G \in B^2_{\bar{\alpha}(n)}$. This was proved in [3] .

Our next lemma gives a higher type version of this observation.

Lemma 4

Let $\phi \in Ct(k)$, $\{e\}(\phi) \simeq s$ by S1 - S9. Then there is a Σ^1_{k-2}-set $A \subseteq H$ such that if $\phi(F)$ is used in a subcomputation of $\{e\}(\phi)$ then $h_F \in A$.

Proof

Let α be an associate for ϕ . Then the following set C will be $\underset{\sim}{\Sigma}^1_{k-2}(\alpha)$:

$$C = \{ \langle d,\vec{f},\alpha,\vec{g},t\rangle \text{ ; each } f_i, g_j \text{ are associates for functionals } G_i, T_j \text{ of type } \leq k-2 \text{ and } \{d\}(\vec{G},\phi,\vec{T})\simeq t \text{ is a subcomputation of } \{e\}(\phi)\} .$$

From C it is easy to construct A as we want.

Lemma 5

Let $\{e\}(\phi) \simeq s$. Let α be a semi-associate for ϕ such that whenever $\phi(F)$ is used in a subcomputation of $\{e\}(\phi)$ then α secures all associates for F. Then there is an n such that

$$\forall\psi \in B^k_{\bar{\alpha}(n)} \quad (\{e\}(\psi)\downarrow \Rightarrow \{e\}(\psi) \simeq s)$$

Proof

The standard proof used when α is an associate will work in this case too.

Remark

Lemmas 4 and 5 may easily be proved for a list $\vec{\phi}$ of arguments instead of just for ϕ .

3. The construction

The strategy now is as follows

1. We construct a recursively compact set K such that

   i All $\beta \in K$ are semi-associates for ${}^k0$.

<u>ii</u> No $\beta \in K$ is an associate.

<u>iii</u> If $A \subseteq H$ is Σ^1_{k-2} then there is a $\beta \in K$ such that if $h_F \in A$ then β secures all associates for F .

2. For each ϕ we construct a sequence δ^{ϕ}_m uniformly primitive recursive in ϕ such that $\lim_{m\to\infty} \delta^{\phi}_m$ will be the principal associate for ϕ .

3. We show that if
$$\Delta_K(\phi) = \mu n \ \forall m \geq n \ \forall \beta \in K \ (\ \bar{\beta}(m) \neq \delta^{\phi}_m \ )$$
then Δ_K has a recursive associate.

4. If $\forall\phi(\ \Delta_K(\phi) = \{e\}(\phi,\psi) \ )$ then by lemma 4 and lemma 5 there will be a $\beta \in K$ such that $\Delta_K({}^k0)$ is determined by a finite part $\bar{\beta}(n)$ of β . We will show that this is as absurd as it seems.

<u>Remark</u>

1 - 4 give the main idea behind the construction. In order to carry through the technical arguments we must choose both K and δ^{ϕ}_m with some care and define Δ_K in a slightly different way.

From now on let $\Sigma(\alpha,h)$ be the following relation
$$\Sigma(\alpha,h) \Leftrightarrow \exists B \in \Sigma^1_{k-2}(\alpha) \ (\ B \subseteq H \ \wedge \ h \in B \ )$$
Then Σ is Π^1_{k-1} and by lemma 3.b there is a primitive recursive relation R such that
$$\Sigma(\alpha,h_1) \Leftrightarrow \forall h_2 \in H \ \exists n \ R(\bar{\alpha}(n),\bar{h}_1(n),\bar{h}_2(n),n)$$
For each σ let

$$\sigma_i(\delta) = \begin{cases} (\sigma(\delta)-1)_i+1 & \text{if } \sigma(\delta) > 0 \\ 0 & \text{if } \sigma(\delta) = 0 \end{cases} \qquad i \in \{1,2\}$$

where $(\ )_1$ and $(\ )_2$ are the two projection maps of the standard pairing operator $<\ ,\ >$.

For each σ we let h_σ be the largest sequence such that

$h_\sigma(i) = s$ if $\exists\delta(\sigma(\delta) = s+1 \wedge \xi_i \in B_\delta^{k-2})$

If B_σ^{k-1} contains more than one element then h_σ is a finite sequence uniformly recursive in σ.

Define

$$P_\alpha(\sigma) = \begin{cases} 1 & \text{if } B_\sigma^{k-1} \text{ contains just one element or if} \\ & \exists n\ R(\bar{\alpha}(n), \bar{h}_{\sigma_1}(n), \bar{h}_{\sigma_2}(n), n) \\ 0 & \text{otherwise} \end{cases}$$

P_α is uniformly recursive in α and P_α is a semi-associate for k_0.

Lemma 6

a If $A \subseteq H$ is Σ^1_{k-2} then there is an $\alpha \in \{0,1\}^{\mathbb{N}}$ such that if $h_F \in A$ then α secures all associates for F.

b P_α is not an associate.

c If $P_\alpha(\sigma) = 1$ and $B_\tau^{k-1} \subseteq B_\sigma^{k-1}$ then $P_\alpha(\tau) = 1$.

Proof

a Let $\alpha \in \{0,1\}^{\mathbb{N}}$ be such that A is $\Sigma^1_{k-2}(\alpha)$ Let $B = \{h_1 : h \in A\}$ where $h_1(n) = (h(n))_1$. Then $B \subseteq H$ is $\Sigma^1_{k-2}(\alpha)$ so $\Sigma(\alpha, h_1)$ for all $h \in A$. Let $h_2(n) = (h(n))_2$.

Then for $h \in A$

$$\exists n\ R(\bar{\alpha}(n),\bar{h}_1(n),\bar{h}_2(n),n)$$

Let β be an associate for F, $h_F \in A$. Let $h = h_F$. Then $h_1 = \lim_{m\to\infty} h_{(\bar{\beta}(m))_1}$ and $h_2 = \lim_{m\to\infty} h_{(\bar{\beta}(m))_2}$. It follows that for some m $P_\alpha(\bar{\beta}(m)) = 1$.

b Let α be given. Let $C = \cup\{B \subseteq H : B$ is $\Sigma^1_{k-2}(\alpha)\}$. Then $C \subseteq H$ and C is $\underset{\sim}{\Sigma}^1_{k-2}$. So there is an $h_1 \in H \setminus C$ and then $\neg\Sigma(\alpha,h_1)$. Choose $h_2 \in H$ such that $\forall n \neg R(\bar{\alpha}(n),\bar{h}_1(n),\bar{h}_2(n),n)$. Let $h_1 = h_{F_1}$, $h_2 = h_{F_2}$ and let β be an associate for $F = \langle F_1,F_2\rangle$. It is clear that F_1 cannot be constant (since otherwise $\{h_1\} \in \Sigma^1_{k-2}$) so $B^{k-1}_{\bar{\beta}(n)}$ will always contain more than one element. (If B^{k-1}_σ contains just one element, that element is constant). It follows that P_α will not secure β.

c This is trivial from the following monotonity property:

$$B^{k-1}_\tau \subseteq B^{k-1}_\sigma \Rightarrow h_\sigma \prec h_\tau$$

which again follows trivially from the definition of h_σ and h_τ. (Use lemma 1.a)

This ends the proof of lemma 6.

Let $K_k = \{P_\alpha : \alpha \in \{0,1\}^{\mathbb{N}}\}$

Then K_k is compact and contains only semi-associates for k_0 none of which are associates.

We will now show that from such compact sets K we may construct interesting functionals of type $k+1$.

Definition

Let $\phi \in Ct(k)$. Let δ_m^ϕ be the sequence of length m defined as follows. For $\sigma<m$ let

$$\delta_m^\phi(\sigma) = \begin{cases} s+1 & \text{if } \exists\tau<m\ (\sigma\neq\tau \wedge B_\tau^{k-1} \subseteq B_\sigma^{k-1}) \\ & \wedge\ \forall\tau<m\ (B_\tau^{k-1} \subseteq B_\sigma^{k-1} \Rightarrow \phi(F_\tau) = s) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 7

$\lim_{m\to\infty} \delta_m^\phi$ is the principal associate for ϕ .

The proof is standard.

4. The proof.

Lemma 8

Let K be a compact set of semi-associates for type k functionals such that K contains no associates. Then the functional

$$\Delta_K(\phi) = \mu n\ \forall m\geq n\ \forall\beta\in K\ \exists\sigma<m\ (\beta(\sigma)=0 \wedge \delta_m^\phi(\sigma)>0\ )$$

is well-defined and has an associate recursive in K , i.e. in

$$\{\langle n,\pi_1, \ldots ,\pi_{k_n}\rangle\ :\ \{\pi_1, \ldots ,\pi_{k_n}\} = \{\bar\beta(n)\ :\ \beta\in K\}\}$$

Proof

Let α be an associate for ϕ . It is sufficient to show that $\Delta_K(\phi)$ is uniformly recursive in α, K .

For each β , if β is a semi-associate and

$$\forall\sigma\ (\alpha(\sigma) > 0 \Rightarrow \beta(\sigma) > 0\ )$$

then β is an associate. So

$$\forall\beta \in K\ \exists\sigma\ (\beta(\sigma)=0 \wedge \alpha(\sigma)>0\ )$$

Since K is compact we may choose these σ's among a finite set

$\{\sigma_1, \ldots, \sigma_k\}$. Choose m so large that all these sequences have proper extensions $<m$. Then

$$\forall\beta\in K\ \exists\sigma<m\ \exists\tau<m\ (\sigma\neq\tau \wedge B_\tau^{k-1} \subseteq B_\sigma^{k-1} \wedge \beta(\sigma)=0 \wedge \alpha(\sigma)>0)$$

Recursively in α, K we may pick m_0 to be the least such m . We then know that

$$\forall m\geq m_0\ \forall\beta\in K\ \exists\sigma<m\ (\beta(\sigma)=0 \wedge \delta_m^\phi(\sigma)>0\ )$$

Then $\Delta_K(\phi)$ is the least $n\leq m_0$ such that

$$\forall m\ (n\leq m\leq m_0 \Rightarrow \forall\beta\in K\ \exists\sigma<m\ (\beta(\sigma)=0 \wedge \delta_m^\phi(\sigma)>0))$$

We may find this n uniformly recursive in K, m_0 . This shows that Δ_K is recursive in K and ends the proof of lemma 8.

Let $\Delta_k = \Delta_{K_k}$. Then $\Delta_k \in Ct(k+1)$ has a recursive associate.

<u>Lemma 9</u>

Δ_k is not Kleene-computable in any $\psi \in Ct(k)$.

<u>Proof</u>

Assume that the lemma is false. Then there is a $\psi \in Ct(k)$ and an e such that

$$\forall\phi \in Ct(k)\ (\ \Delta_k(\phi) = \{e\}(\phi,\psi))\ .$$

By lemma 4 there is a $\underset{\sim}{\Sigma}_{k-2}^1$-set $A \subseteq H$ such that whenever ${}^kO(F)$ is used in a subcomputation of $\{e\}({}^kO,\psi)$ then $H_F \in A$. By lemma 6.a there is a $P_\alpha \in K_k$ securing all associates for F whenever $h_F \in A$. By lemma 5 there is an n such that whenever $\phi \in B^k_{\bar{P}_\alpha(n)}$ then

$$\Delta_k(\phi) = \{e\}(\phi,\psi) = \{e\}({}^kO,\psi) = \Delta_k({}^kO)$$

We defined δ_m^{ϕ} for $\phi \in Ct(k)$ but we can use the same definition for all ϕ defined on all F_σ . Let

$$\phi_0(F_\sigma) = \begin{cases} 0 & \text{if } B_\sigma^{k-1} \text{ contains just one element} \\ & \text{or if } \exists\tau<n\ (F_\sigma \in B_\tau^{k-1} \wedge P_\alpha(\tau)=1) \\ \sigma+1 & \text{otherwise} \end{cases}$$

By lemma 2.ii we see that ϕ_0 is well-defined. Moreover, if $\forall\phi \in B^k_{\bar{P}_\alpha(n)}\ (\phi(F_\sigma)=s)$ then $\phi_0(F_\sigma)=s$, so all finite parts of ϕ_0 may be extended to elements in $B^k_{\bar{P}_\alpha(n)}$.

<u>Claim</u>

<u>a</u> $\forall m>n\ \forall\sigma\ (n\leq\sigma<m \Rightarrow \delta_m^{\phi_0}(\sigma) \leq P_\alpha(\sigma)\ )$

<u>b</u> If $\sigma<n$ and $P_\alpha(\sigma)=0$ then there is an m_0 such that $m\geq m_0 \Rightarrow \delta_m^{\phi_0}(\sigma)=0$

<u>Proof</u>

For each m, σ we have that $\delta_m^{\phi_0}(\sigma)$ is either 0 or $\phi_0(F_\sigma)+1$, and $\phi_0(F_\sigma)$ is either 0 or $\sigma+1$.

If $\delta_m^{\phi_0}(\sigma)=\sigma+2$ then

$$\exists\sigma_1<m\ (\sigma_1\neq\sigma \wedge B_{\sigma_1}^{k-1} \subseteq B_\sigma^{k-1} \wedge \phi_0(F_\sigma)=\phi_0(F_{\sigma_1})=\sigma+1)$$

But $\phi_0(F_{\sigma_1})\neq\sigma+1$ when $\sigma\neq\sigma_1$ so this is impossible. It follows that $\delta_m^{\phi_0}(\sigma) \in \{0,1\}$ for all σ .

<u>a</u> Assume that $n\leq\sigma<m$ and $\delta_m^{\phi_0}(\sigma)=1$. Since $\delta_m^{\phi_0}(\sigma)>0 \Rightarrow \delta_m^{\phi}(\sigma)=\phi(F_\sigma)+1$ for all ϕ, σ we must have $\phi_0(F_\sigma)=0$. If this is because B_σ^{k-1} contains just one element we have

constructed P_α in such a way that $P_\alpha(\sigma)=1$.

If B_σ^{k-1} contains more than one element we must have

$$\exists\tau<n\ (F_\sigma \in B_\tau^{k-1} \wedge P_\alpha(\tau)=1\ )$$

Then $\tau<\sigma$ and by lemma 2.iii we must have $B_\sigma^{k-1} \subseteq B_\tau^{k-1}$.

But then by lemma 6.c. $P_\alpha(\sigma)=1$.

So $\delta_m^{\phi_0}(\sigma)=1 \Rightarrow P_\alpha(\sigma)=1$

b If $P_\alpha(\sigma)=0$ then B_σ^{k-1} contains more than one element. If $B_\sigma^{k-1} \subseteq \cup\ \{B_\tau^{k-1} : \tau<n \wedge P_\alpha(\tau)=1\}$ then by lemma 1.b.

$$\exists\tau<n\ (P_\alpha(\tau)=1 \wedge B_\sigma^{k-1} \subseteq B_\tau^{k-1})\ .$$

But by lemma 6.c. $P_\alpha(\sigma)=1$ so this is impossible. So $B_\sigma^{k-1} \not\subseteq \cup\ \{B_\tau^{k-1} : \tau<n \wedge P_\alpha(\tau)=1\}$. By lemma 1.c. there are extensions σ_1 and σ_2 of σ such that $\sigma_1 \not\preceq \sigma_2$ and

$$B_{\sigma_1}^{k-1} \cap \cup\ \{\ B_\tau^{k-1} : \tau<n \wedge P_\alpha(\tau)=1\} = \emptyset\ .$$

Then $\phi_0(F_{\sigma_1}) = \sigma_1+1$ and $\phi_0(F_{\sigma_2}) = \sigma_2+1$. For $m>\sigma_2$ we see that $\delta_m^{\phi_0}(\sigma)=0$. This ends the proof of the claim.

By the claim we have

$$\exists m_0>n\ \forall m\geq m_0\ \forall\sigma< m\ (\delta_m^{\phi_0}(\sigma) \leq P_\alpha(\sigma)).$$

Choose $m > \max\{\Delta_K({}^k0),m_0\}$. Let $\phi \in B^k_{\bar{P}_\alpha(n)}$ be such that

$$\forall\sigma<m\ \phi(F_\sigma) = \phi_0(F_\sigma)\ .$$

As we remarked after the definition of ϕ_0 this is possible.

Then $\delta_m^\phi=\delta_m^{\phi_0}$ and $\forall\sigma<m\ \delta_m^\phi(\sigma)\leq P_\alpha(\sigma)$. So $\Delta_k(\phi)\geq m$. But since

$\phi \in B^k_{\bar{p}_\alpha}(n)$ we have $\Delta_k(\phi) = \Delta_k({}^k0)$. This is a contradiction and the lemma is proved.

We have now showed

Theorem

For each $k \geq 2$ there is a recursive functional Δ in $Ct(k+1)$ such that Δ is not computable in any functional in $Ct(k)$.

Proof

For $k=2$ we may use the fan-functional while for $k \geq 3$ we have showed that Δ_k is an example.

5. References

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